Smooth Morse-Lyapunov Functions and Morse Theory of Strong Attractors for Nonsmooth Dynamical Systems [1]

Desheng Li

Department of Math., Tianjin University
Tianjin 300072, P.R. China
Email: lidsmath@tju.edu.cn, lidsmath@hotmail.com

Date: May 18, 2010

Abstract. In this paper we first construct smooth Morse-Lyapunov functions of attractors for nonsmooth dynamical systems. Then we prove that all open attractor neighborhoods of an attractor have the same homotopy type. Based on this basic fact we finally introduce the concept of critical group for Morse sets of an attractor and establish Morse inequalities and equations.

Contents

[1] This work is supported by NNSF of China (10771159) and NSF of Tianjin (09JCYBJC01800).

2010 *Mathematics Subject Classification.* 37B25, 37B30.

Key words and phrases. Nonsmooth dynamical system, attractor, Morse-Lyapunov function, Morse theory.

1 Introduction

Attractors are of particular interest in the theory of dynamical systems, this is because that much of the longtime dynamics of a given system is represented by the dynamics on and near the attractors.

The attractors of smooth dynamical systems have been extensively studied in the past decades, in both finite and infinite dimensional cases. The existence results (especially for infinite dimensional systems) are well known; see [12, 33, 49, 53] and [56] etc. In many cases one can give an estimate on the Hausdorff (or fractional, or informational) dimension of an attractor. It can even be proved that the global attractor of an infinite dimensional system is actually contained in a finite-dimensional manifold; see [17, 56] for details. The Morse theory is also fully developed [15, 16, 27, 31, 47, 48, 50, 52]. In contrast, the situation in nonsmooth dynamical systems seems to be more complicated, and by far some fundamental problems concerning attractors are still undergoing investigations [8, 9, 11, 28, 36, 43, 44].

Nonsmooth dynamical systems appear widely in a large variety of applications such as mechanics with dry friction, electric circuits with small inductivity, systems with small inertial, economy, biology, viability theory, control theory, game theory and optimization etc. [1, 2, 4, 5, 6, 14, 18, 19, 20, 21, 22, 24, 26, 35, 57]. The rapid growth of such systems in recent years challenges mathematicians to develop more direct and uniform approaches to study their dynamics. In this paper we are basically interested in finite dimensional case, in which nonsmooth systems can be typically described in a quite uniform manner via the following differential inclusion:

$$x'(t) \in F(x(t)), \qquad x(t) \in X := \mathbb{R}^m. \tag{1.1}$$

One of the main feature of (1.1) is that it may fail to have uniqueness on solutions. Because of this, it usually generates a multi-valued semiflow. So one needs to distinguish dynamical concepts between strong and weak sense, where the former means that they apply to "all" solutions, and in the latter "all" is replaced by "some". The two settings are rather different. We make precise that in this present work we will be solely interested in the *strong* case. Hence from now on all the dynamical concepts concerning (1.1) should be understood in the strong sense, except otherwise statement.

In our previous work [36] we have discussed Morse decompositions of attractors for (1.1). Morse decompositions reveal some topological structures of attractors, and are of crucial importance in the understanding of the dynamics inside attractors. Here we want go a further step. First we construct smooth Morse-Lyapunov functions for attractors. Then we prove that all open attractor neighborhoods of an attractor have the same homotopy type. Based on this basic fact we further introduce the concept of critical groups for Morse sets and establish Morse inequalities and equations of attractors.

Now let us give a brief description of what we will do. We will assume and *only* assume throughout the paper that F satisfies the following **standing assumptions**:

(H1) $F(x)$ is a nonempty convex compact subset of X for every $x \in X$;

(H2) $F(x)$ is upper semi-continuous in x.

Let $\mathscr{A}$ be an attractor of (1.1) with attraction basin $\Omega = \Omega(\mathscr{A})$ and Morse decomposition $\mathcal{M} = \{M_1, \cdots, M_l\}$. We will construct a radially unbounded smooth function $V \in C^\infty(\Omega)$ such that

(1) V is constant on each Morse set M_k, and

(2) V is strictly decreasing along solutions of (1.1) in Ω outside the Morse sets.

Moreover, there is a nonnegative function $w \in C(\Omega)$ which is positive on $\Omega \setminus \mathcal{D}$, where $\mathcal{D} = \cup_{1 \le k \le l} M_k$, such that

$$\max_{v \in F(x)} \nabla V(x) \cdot v \le -w(x), \qquad \forall\, x \in \Omega \setminus \mathcal{D}. \tag{1.2}$$

Lyapunov functional characterizations of dynamical behavior are usually known as converse Lyapunov theorems. This question can be traced back to Lyapunov [41]. One of the early important milestones in the pursuit of smooth converse Lyapunov functions was Massera's 1949 paper [42]. Since then a vast body of fundamental results and techniques are obtained for smooth dynamical systems (and even for ODE systems with only continuous righthand sides [32]); see references cited in [13, 57] etc. Nevertheless, the problem of smooth Lyapunov characterization of stability for nonsmooth systems had remained open for a long time, although Krasovskii pointed out as early as 1959 the desirability of such characterizations [29]. A great progress in nonsmooth case was first made by Clarke, Ledyaev and Stern in their 1998 paper [13], in which the authors proved a smooth converse Lyapunov theorem for strong global asymptotic stability of the zero solution of (1.1). Later on a more general framework appeared in Tell and Praly's work [57]. Other related works can be found in [40, 54] etc.

Note that an attractor $\mathscr{A}$ always has a trivial Morse decomposition $\mathcal{M} = \{\mathscr{A}\}$. In such a case a Morse-Lyapunov function of $\mathcal{M}$ reduces to a Lyapunov one of the attractor itself. In particular, in case $\mathscr{A}$ is an equilibrium we recover the converse Lyapunov theorem in [13], except that we don't require the function w in (1.2) to be smooth. To the best of our knowledge, the above smooth converse Lyapunov theorem for Morse decompositions is rather new even if we come back to the situation of smooth dynamical systems.

We then try to establish a Morse theory of attractors for nonsmooth dynamical systems by employing smooth Lyapunov functions and Morse-Lyapunov functions. In the smooth case this can be done by using Conley index or shape theory [15, 16, 27, 31, 48, 50, 51]. Here we want to develop a somewhat different approach for

nonsmooth systems. First, we prove that all the open attractor neighborhoods of an attractor have the same homotopy type. Then based on this basic fact we introduce the concept of critical group for Morse sets and establish Morse equations and inequalities. Specifically, let

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_l = \mathscr{A}$$

be the Morse filtration of the Morse decomposition $\mathcal{M} = \{M_1, \cdots, M_l\}$. We define the critical group $C_*(M_k)$ of a Morse set M_k to be the homologies of the space pair (W, U) (with coefficients in a given Abelian group $\mathcal{G}$) for any open attractor neighborhoods W of A_k and U of A_{k-1}, that is,

$$C_*(M_k) = H_*(W, U).$$

This definition is independent of the choice of W and U. Set

$$\mathfrak{m}_q = \sum_{1 \le k \le l} \operatorname{rank} C_q\left(M_k\right), \qquad q = 0, 1, \cdots.$$

Let $\beta_q := \beta_q(\Omega) = \operatorname{rank} H_q(\Omega)$ be the q-th Betti number of the attraction basin Ω. We prove that the following Morse inequalities and equation hold:

$$\mathfrak{m}_0 \ge \beta_0,$$

$$\mathfrak{m}_1 - \mathfrak{m}_0 \ge \beta_1 - \beta_0,$$

$$\cdots\cdots$$

$$\mathfrak{m}_m - \mathfrak{m}_{m-1} + \cdots + (-1)^m \mathfrak{m}_0 = \beta_m - \beta_{m-1} + \cdots + (-1)^m \beta_0.$$

An alternative approach to establish Morse theory for attractors of nonsmooth systems might be that one can still use Conley index (which will not be developed here). The interested reader is referred to Kunze, Kupper and Li [30] and Mrozek [46] etc. for Conley index theory of nonsmooth systems.

2 Preliminaries

Let $X = \mathbb{R}^m$, which is equipped with the usual norm $|\cdot|$. For convenience in statement, we will identify a single point $a \in X$ with the singleton $\{a\}$.

For any nonempty subsets A and B of X, define the *Hausdorff semi-distance* and *Hausdorff distance*, respectively, as

$$d_{\mathrm{H}}(A, B) = \sup_{x \in A} d(x, B), \qquad \delta_{\mathrm{H}}(A, B) = \max\left\{d_{\mathrm{H}}(A, B), d_{\mathrm{H}}(B, A)\right\},$$

where $d(x, B) = \inf_{y \in B} |x - y|$. We also assign $d_{\mathrm{H}}(\emptyset, B) = 0$.

The *closure* of A is denoted by $\overline{A}$, and the *interior* and *boundary* of A are denoted by $\mathrm{int}A$ and ∂A, respectively. We use $\mathrm{B}(A,r)$ to denote the *r-neighborhood* of A, i.e.,

$$\mathrm{B}(A,r)=\{y\in X|\ \ d(y,A)<r\}.$$

In particular, $\mathrm{B}_r=\mathrm{B}(0,r)$ is the *ball* in X centered at 0 with radius r.

We say that a subset V of X is a *neighborhood* of A, this means $\overline{A}\subset \mathrm{int}\,V$.

2.1 Some basic facts on differential inclusions

Let I be an interval. A map $x(\cdot):I\rightarrow X$ is said to be a *solution* of (1.1) on I, if it is absolutely continuous on any compact interval $J\subset I$ and solves (1.1) at a.e. $t\in I$.

A solution on $\mathbb{R}^1$ will be simply called a *complete solution*.

Let $x\in X$, and $A\subset X$. We denote by $\mathcal{S}_x$ the family of solutions $x(\cdot)$ of (1.1) with initial value $x(0)=x$, and $\mathcal{S}_A=\cup_{x\in A}\mathcal{S}_x$. Define set-valued map $\mathcal{R}$ on $\mathbb{R}^+\times X$ as:

$$\mathcal{R}(t)x=\{x(t)|\ x(\cdot)\in\mathcal{S}_x \text{ which exists on } [0,t]\}\,,\qquad \forall\,(t,x)\in\mathbb{R}^+\times X.$$

Then $\mathcal{R}$ satisfies the following semigroup property:

$$\mathcal{R}(0)x=x,\qquad \forall\,x\in X,$$

$$\mathcal{R}(s)\mathcal{R}(t)x=\mathcal{R}(s+t)x,\qquad \forall\,s,t\geq 0,\ x\in X.$$

We will call $\mathcal{R}$ the *multi-valued semiflow* (or *set-valued semiflow*) generated by (1.1), sometimes written as $\mathcal{R}(t)$.

Lemma 2.1 *[3, 19]. Let I be a compact interval. Then, for any sequence $\delta_n\rightarrow 0$ and sequence $x_n(\cdot)$ of absolutely continuous and uniformly bounded functions on I satisfying:*

$$x_n'(t)\in\overline{con}\ F\left(x_n(t)+\delta_n\overline{B}_1\right)+\delta_n\overline{B}_1,$$

there exists a subsequence $x_{n_k}(\cdot)$ converging uniformly to a solution $x(\cdot)$ of (1.1) on I.

Lemma 2.2 *[38] Let K be a compact subset of X and let $0<T<\infty$. If no solution $x(\cdot)\in\mathcal{S}_K$ blows up on $[0,T]$, then there exists an $R>0$ such that*

$$|x(t)|\leq R,\qquad \forall t\in[0,T],\ x(\cdot)\in\mathcal{S}_K.$$

Consequently $\mathcal{R}([0,T])K$ is compact.

2.2 Dynamical concepts and attractors in the strong sense

As we emphasized in the introduction, all the dynamical concepts in this work should be understood *in the strong sense*, except otherwise statement.

Let $A, B \subset X$. We say that A *attracts* B, this means that no solution $x(\cdot) \in \mathcal{S}_B$ blows up in finite time, moreover,

$$\lim_{t\to+\infty} d_{\mathrm{H}}(\mathcal{R}(t)B,\, A) = 0.$$

The *attraction basin* $\Omega(A)$ of A is defined as:

$$\Omega(A) = \{x \in X|\ \ A \text{ attracts } x\}.$$

The set A is said to be *positively invariant* (resp. *invariant*), if

$$\mathcal{R}(t)A \subset A \ \ (\text{resp. } \mathcal{R}(t)A = A\,), \qquad \forall\, t \geq 0.$$

A is said to be *weakly invariant*, if for any $x \in A$, there passes through x a complete solution $x(\cdot)$ with $x(\mathbb{R}^1) \subset A$. The ω*-limit sets* $\omega(A)$ is defined as:

$$\omega(A) := \{y \in X :\ \exists t_n \to \infty \text{ and } y_n \in \mathcal{R}(t_n)A \text{ such that } y_n \to y\},$$

Definition 2.3 *Let $\mathscr{A}$ be a compact subset of X. If there is a neighborhood U of $\mathscr{A}$ such that $\mathscr{A} = \omega(U)$, then we say that $\mathscr{A}$ is an attractor of (1.1) (in the strong sense).*

We allow the empty set $\emptyset$ to be an attractor with $\Omega(\emptyset) = \emptyset$.

A global attractor is an attractor $\mathscr{A}$ with $\Omega(\mathscr{A}) = X$.

Proposition 2.4 [*37*, *38*] *Let $\mathscr{A}$ be an attractor of (1.1). Then*

(1) *$\mathscr{A}$ is invariant.*

(2) *$\mathscr{A}$ is Lyapunov stable, that is, for any $\varepsilon > 0$, one can find a $\delta > 0$ such that*

$$\mathcal{R}(t)\mathrm{B}(\mathscr{A}, \delta) \subset \mathrm{B}(\mathscr{A}, \varepsilon), \qquad \forall\, t \geq 0.$$

(3) *$\Omega(\mathscr{A})$ is a positively invariant open neighborhood of $\mathscr{A}$.*

(4) *$\mathscr{A}$ attracts any compact subset K of $\Omega(\mathscr{A})$.*

Concerning the existence of attractors, it is known that if there is a compact set K that attracts a neighborhood of itself, then (1.1) has an attractor $\mathscr{A} \subset K$ (see, e.g., [38]).

Let there be given an attractor $\mathscr{A}$ of (1.1).

Definition 2.5 *An* attractor neighborhood *$\mathcal{O}$ of $\mathscr{A}$ means a positively invariant neighborhood of $\mathscr{A}$ with $\mathcal{O} \subset \Omega(\mathscr{A})$.*

Let $\mathscr{A} \subset U \subset X$. We define the *attraction basin* $\Omega^U(\mathscr{A})$ of $\mathscr{A}$ in U as:

$$\Omega^U(\mathscr{A}) = \Omega(\mathscr{A}) \cap \{x \in U \mid \mathcal{R}(t)x \subset U \text{ for all } t \geq 0\}. \tag{2.1}$$

One easily checks the validity of the following fact by using Lyapunov stability of $\mathscr{A}$.

Proposition 2.6 *Let U be an open neighborhood of $\mathscr{A}$. Then $\Omega^U(\mathscr{A})$ is an open attractor neighborhood of $\mathscr{A}$.*

Now we consider for $\delta \geq 0$ the inflated system of (1.1):

$$x'(t) \in \overline{\mathrm{con}}\ F\left(x(t) + \delta\overline{\mathrm{B}}_1\right) + \delta\overline{\mathrm{B}}_1. \tag{2.2}$$

The proof of the robustness result below can be obtained by slightly modifying the one for Theorem 2.10 in [39] (see also Theorem 2.9 in [36]). We omit the details.

Theorem 2.7 *Let $\mathscr{A}$ be an attractor of (1.1). Then there exists a $\delta_0 > 0$ such that the inflated system (2.2) has an attractor $\mathscr{A}(\delta)$ for $\delta < \delta_0$. Moreover, it holds that*

(1) $\delta_{\mathrm{H}}\left(\mathscr{A}(\delta), \mathscr{A}\right) \to 0$ *as* $\delta \to 0$*;*

(2) *for any open neighborhood U and compact set $K \subset \Omega^U(\mathscr{A})$, we have*

$$K \subset \Omega^U(\mathscr{A}(\delta)),$$

provided δ is sufficiently small.

2.3 Morse decompositions of attractors

Let $\mathscr{A}$ be an attractor of (1.1) with attraction basin $\Omega = \Omega(\mathscr{A})$.

We say that a compact subset A of $\mathscr{A}$ is an *attractor* in $\mathscr{A}$, this means that there exists a neighborhood U of A such that

$$\omega\left(U \cap \mathscr{A}\right) = A.$$

Theorem 2.8 [36] *Let $A \subset \mathscr{A}$ be an attractor in $\mathscr{A}$. Then A is also an attractor of (1.1) in X.*

Let A be an attractor in $\mathscr{A}$. Define

$$A^* = \{x \in \mathscr{A} \mid \omega(x) \setminus A \neq \emptyset\}.$$

A^* is said to be a *repeller* of (1.1) in $\mathscr{A}$ dual to A, and (A, A^*) is said to be an *attractor-repeller pair* in $\mathscr{A}$.

It is known that A^* is compact and weakly invariant [36]. Moreover, we have

$$A^* = \mathscr{A} \setminus \Omega^{\mathscr{A}}(A) = \mathscr{A} \setminus \Omega(A). \tag{2.3}$$

Definition 2.9 *Let $\mathscr{A}$ be an attractor of (1.1). An ordered collection $\mathcal{M} = \{M_1, \cdots, M_l\}$ of compact subsets of $\mathscr{A}$ is called a Morse decomposition of $\mathscr{A}$, if there exists an increasing sequence*

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_l = \mathscr{A} \tag{2.4}$$

of attractors in $\mathscr{A}$, called Morse filtration of $\mathcal{M}$, such that

$$M_k = A_k \cap A_{k-1}^*, \qquad 1 \le k \le l. \tag{2.5}$$

The sets M_k are called Morse sets.

Remark 2.10 *Since a Morse set M_k may disappear under perturbations, for convenience in statement, we allow M_k to be the empty set $\emptyset$. This occurs in case $A_k = A_{k-1}$. However, if two Morse decompositions $\mathcal{M}$ and $\mathcal{M}'$ have the same nonempty Morse sets, they will be regarded as the same.*

The following result taken from [36] provides some information on Morse decompositions of attractors.

Theorem 2.11 *Let $\mathcal{M} = \{M_1, \cdots, M_l\}$ be a Morse decomposition of $\mathscr{A}$ with Morse filtration $\emptyset = A_0 \subset A_1 \subset \cdots \subset A_l = \mathscr{A}$. Then*

(1) *For each k, (A_{k-1}, M_k) is an attractor-repeller pair in A_k;*

(2) *M_k are pair-wise disjoint weakly invariant compact sets;*

(3) *If γ is a complete trajectory, then either $\gamma(\mathbb{R}) \subset M_k$ for some Morse set M_k, or else there are indices $i < j$ such that $\alpha(\gamma) \subset M_j$ and $\omega(\gamma) \subset M_i$;*

(4) *The attractors A_k are uniquely determined by the Morse sets, that is,*

$$A_k = \bigcup_{1 \le i \le k} W^u(M_i), \qquad 1 \le k \le l,$$

where $W^u(M_i)$ is the unstable manifold of M_i, namely,

$$W^u(M_i) = \{x |\ \text{there is a trajectory } \gamma : \mathbb{R} \to \mathscr{A} \text{ through } x \text{ with } \alpha(\gamma) \subset M_i\}.$$

3 Smooth Lyapunov Functions of Attractors

Let U be an open subset of X.

A nonnegative function $\alpha \in C(U)$ is said to be *radially unbounded* on U, notated by $\alpha \in \mathcal{K}^\infty$, if for any $R > 0$ there exists a compact subset $K \subset U$ such that

$$\alpha(x) > R, \qquad \forall\, x \in U \setminus K.$$

A continuous function V on an open neighborhood U of an attractor $\mathscr{A}$ is said to be a *Lyapunov function* of $\mathscr{A}$ on U, if

(1) V is constant on $\mathscr{A}$;

(2) $V(x(t))$ is strictly decreasing in t for any solution $x(\cdot)$ of (1.1) in $U \setminus \mathscr{A}$.

Our main results in this part are the following two theorems. One asserts the existence of bounded smooth Lyapunov functions of attractors defined on the whole phase space X, which will be used to construct Morse-Lyapunov functions of attractors in Section 4. The other relates to the existence of smooth $\mathcal{K}^\infty$ Lyapunov functions of attractors, which plays an important role in discussing topological properties of attractors.

Theorem 3.1 *Let $\mathcal{O}$ be an open attractor neighborhood of $\mathscr{A}$. Then there exists a smooth function $V \in C^\infty(X)$ such that*

$$V|_{\mathscr{A}} = 0, \qquad V|_{\mathscr{A}^c} > 0, \qquad V|_{\mathcal{O}^c} = 1 \tag{3.1}$$

$$\max_{v \in F(x)} \nabla V(x) \cdot v \leq -w(x), \qquad\qquad \forall\, x \in X, \tag{3.2}$$

where $w \in C(X)$ is a nonnegative function satisfying:

$$w|_{\mathscr{A} \bigcup \mathcal{O}^c} = 0, \qquad w|_{\mathcal{O} \setminus \mathscr{A}} > 0. \tag{3.3}$$

Theorem 3.2 *Let $\mathcal{O}$ be an open attractor neighborhood of $\mathscr{A}$. Then there exists a $\mathcal{K}^\infty$ function $V \in C^\infty(\mathcal{O})$ that vanishes on $\mathscr{A}$, such that*

$$\max_{v \in F(x)} \nabla V(x) \cdot v \leq -w(x), \qquad\qquad \forall\, x \in \mathcal{O}, \tag{3.4}$$

where $w \in C(\mathcal{O})$ is a nonnegative function satisfying:

$$w|_{\mathscr{A}} = 0, \qquad w|_{\mathcal{O} \setminus \mathscr{A}} > 0. \tag{3.5}$$

A smooth converse Lyapunov theorem for strongly asymptotically stable compact sets of (1.1) in case $F(x)$ is continuous in x can be found in [54] (the approach therein is mainly based on some arguments which seem to be more geometrical).

In what follows we first construct smooth Lyapunov functions when F is locally Lypschitz. Then we prove Theorem 3.1 by appropriate approximations and some smoothing method used in [13, 40] etc. Theorem 3.2 follows directly from Theorem 3.1.

We remark that although we follow the procedure in [13] and [40] etc., our construction method of the Lyapunov functions here is quite different from those in the literature. In contrast, ours seems to be more direct and simpler, and can be easily handled.

3.1 Locally Lipschitz F vs locally Lipschitz Lyapunov function

In this subsection we assume that F is locally Lipschitz on X, that is, for any compact subset $K \subset X$, there exists an $L > 0$ such that

$$F(x) \subset F(y) + L|x-y|\,\overline{\mathrm{B}}_1, \qquad \forall\, x, y \in K. \tag{3.6}$$

Let U be an open subset of X. For $V \in C(U)$, $x \in U$ and $v \in X$, define

$$D_v^+ V(x) = \limsup_{\tau \to 0^+} \frac{V(x+\tau v) - V(x)}{\tau}.$$

$D_v^+ V(x)$ is called the *Dini supderivative* of V along the vector v.

The main result in this subsection is contained in the following proposition.

Proposition 3.3 *Let $\mathcal{O}$ be an open attractor neighborhood of $\mathscr{A}$. Then there exists on $\mathcal{O}$ a locally Lipschitz $\mathcal{K}^\infty$ function V vanishing on $\mathscr{A}$ such that*

$$\max_{v \in F(x)} D_v^+ V(x) \le -w(x), \qquad \forall\, x \in \mathcal{O}, \tag{3.7}$$

where $w \in C(\mathcal{O})$ is a nonnegative function satisfying (3.5).

The following lemma plays a basic role in the proof of Proposition 3.3.

Lemma 3.4 *For any $\delta > 0$ with $\overline{B}(\mathscr{A}, \delta) \subset \mathcal{O}$, there exists a locally Lipschitz $\mathcal{K}^\infty$ function $V \in C(\mathcal{O})$ such that*

$$V|_{\mathscr{A}} = 0, \quad V|_{\mathcal{O} \setminus \overline{\mathrm{B}}(\mathscr{A},\delta)} > 0, \tag{3.8}$$

$$\max_{v \in F(x)} D_v^+ V(x) \le -V(x), \qquad x \in \mathcal{O}. \tag{3.9}$$

Proof. Pick a sequence of compact subsets K_n of $\mathcal{O}$ ($n = 0, 1, 2 \cdots$) such that

$$\overline{\mathrm{B}}(\mathscr{A}, \delta) \subset \mathrm{int} K_0 \subset K_0 \subset \cdots \subset K_n \subset \mathrm{int} K_{n+1} \cdots, \quad \mathcal{O} = \cup_{n \ge 0} K_n.$$

Then since the distance between ∂K_n and K_{n-1} is positive, there exists for each n a $\tau_n > 0$ such that

$$\mathcal{R}(t)\partial K_n \subset \mathcal{O} \setminus K_{n-1}, \qquad t \in [0, \tau_n). \tag{3.10}$$

Take a nonnegative function $a_0 \in C^\infty(X)$ with

$$a_0(x) \equiv 0 \ \ (\text{on } \mathrm{B}(\mathscr{A}, \delta/2)\,), \quad \text{and } a_0(x) > 0 \ \ (\text{on } X \setminus \overline{\mathrm{B}}(\mathscr{A}, \delta)\,). \tag{3.11}$$

For each $n \ge 2$ we choose a nonnegative function $a_n \in C^\infty(X)$ with

$$a_n(x) = \begin{cases} 1, & x \in K_n \setminus K_{n-1}, \\ 0, & x \in K_{n-2} \cup K_{n+1}^c. \end{cases}$$

(Such functions can be easily obtained by appropriately smoothing some continuous ones.) Let

$$\alpha(x) = a_0(x) + \sum_{n=2}^{\infty} \frac{n}{\tau_n} a_n(x). \tag{3.12}$$

Note that the righthand side of (3.12) is actually a finite sum over n. Consequently $\alpha \in C^\infty(X)$. We observe that

$$\alpha(x) \equiv 0 \ \ (\text{on } \overline{\mathrm{B}}(\mathscr{A}, \delta/2)), \qquad \alpha(x) > 0 \ \ (\text{on } \overline{\mathrm{B}}(\mathscr{A}, \delta)^c).$$

Define $V(x)$ as:

$$V(x) = \sup_{x(\cdot) \in \mathcal{S}_x} \int_0^\infty e^t \alpha(x(t))\, dt.$$

For each K_n, since $\mathscr{A}$ attracts K_n, there exists a $T > 0$ such that

$$\mathcal{R}(t) K_n \subset \mathrm{B}(\mathscr{A}, \delta/2), \qquad t > T. \tag{3.13}$$

This implies that V is well defined.

One easily verifies by construction of α that

$$V(x) \equiv 0 \ \ (\text{on } \mathscr{A}), \qquad V(x) > 0 \ \ (\text{on } \mathcal{O} \setminus \overline{\mathrm{B}}(\mathscr{A}, \delta)).$$

We show that $V(x)$ is a $\mathcal{K}^\infty$ function on $\mathcal{O}$. Let $x \in \mathcal{O} \setminus K_n$ ($n \geq 2$). Take a solution $x(\cdot) \in \mathcal{S}_x$, and let

$$s_n = \sup\{t > 0 |\ x([0, t)) \subset \mathcal{O} \setminus K_{n-1}\}.$$

Then $x(s_n) \in K_{n-1}$. By (3.10) one necessarily has

$$x(t) \in K_n \setminus K_{n-1}, \qquad t \in [s_n - \tau_n, s_n),$$

Therefore

$$V(x) \geq \int_{s_n - \tau_n}^{s_n} \alpha(x(t))\, dt \geq \frac{n}{\tau_n} \int_{s_n - \tau_n}^{s_n} a_n(x(t))\, dt = n,$$

and the conclusion follows.

We now prove that V is locally Lipschitz.

Let $x \in K_n$. We claim that there exists a $x_*(\cdot) \in \mathcal{S}_x$ such that

$$V(x) = \int_0^\infty e^t \alpha(x_*(t))\, dt. \tag{3.14}$$

Indeed, by (3.13) we find that

$$V(x) = \sup_{x(\cdot) \in \mathcal{S}_x} \int_0^T e^t \alpha(x(t))\, dt.$$

Let $x_k(\cdot) \in \mathcal{S}_x$ be a sequence such that

$$V(x) = \lim_{k \to +\infty} \int_0^T e^t \alpha(x_k(t))\, dt. \tag{3.15}$$

Then by compactness there exists a subsequence of $x_k(\cdot)$, still denoted by $x_k(\cdot)$, such that $x_k(\cdot)$ converges uniformly on $[0, T]$ to a solution $x_*(\cdot) \in \mathcal{S}_x$. Passing to the limit in (3.15) one immediately obtains the validity of (3.14).

Denote by $g(t, x)$ the unique closest point in $F(x)$ to $x_*'(t)$. Then $g : \mathbb{R}^1 \times X \to X$ is a Carathéodory function (see [19], pp. 49), and therefore solutions of the ODE:

$$y'(t) = g(t, y(t)) \tag{3.16}$$

exist at least locally. For each $y \in K_n$, we pick a solution $y(\cdot)$ of (3.16). Clearly $y(\cdot) \in \mathcal{S}_y$, and hence it exists for all $t \geq 0$. As $\mathcal{R}([0, T])K_n$ is compact, by virtue of (3.13) we deduce that there exists an $N > n$ such that

$$\mathcal{R}(t)K_n \subset K_N, \qquad \forall\, t \geq 0.$$

Since F is locally Lipschitz on X, there is an $L > 0$ such that

$$F(x_1) \subset F(x_2) + L|x_1 - x_2|\overline{\mathrm{B}}_1, \qquad \forall\, x_1, x_2 \in K_N.$$

Thus

$$\begin{aligned} |y'(t) - x_*'(t)| &= |x_*'(t) - g(t, y(t))| = d\,(x_*'(t),\, F(y(t))) \\ &\leq (\text{recall } x_*'(t) \in F(x_*(t))\,) \leq L|x_*(t) - y(t)|, \qquad t \geq 0, \end{aligned}$$

which implies

$$|y(t) - x_*(t)| \leq |x - y|e^{Lt}, \qquad t \geq 0.$$

Therefore

$$\begin{aligned} V(x) &= \textstyle\int_0^\infty e^t \alpha(x_*(t))\, dt = \int_0^T e^t \alpha(x_*(t))\, dt \\ &= \textstyle\int_0^T e^t \alpha(y(t))\, dt + \int_0^T e^t\, (\, \alpha(x_*(t)) - \alpha(y(t))\,)\ dt \\ &\leq V(y) + M|x - y| \textstyle\int_0^T e^{(L+1)t} dt, \end{aligned}$$

where $M > 0$ is the Lipschitz constant of α on K_N. Because $x, y \in K_n$ are arbitrary, we conclude that

$$|V(x) - V(y)| \leq M \int_0^T e^{(L+1)t} dt\, |x - y|, \qquad \forall\, x, y \in K_n.$$

Now let us evaluate $D_v^+ V(x)$ for $x \in \mathcal{O}$ and $v \in F(x)$.

For every y denote by $g(y) \in F(y)$ the unique closest point in $F(y)$ to v. Then the function g is continuous (see [3]), and $g(x) = v$. Let $x(\cdot)$ be a solution to the following initial value problem:

$$x'(t) = g(x(t)), \qquad x(0) = x.$$

Of course $x(\cdot) \in \mathcal{S}_x$. Note that

$$x(\tau) = x + \tau v + o(\tau). \tag{3.17}$$

We write

$$\frac{V(x+\tau v) - V(x)}{\tau} = \frac{V(x(\tau)) - V(x)}{\tau} + \frac{V(x+\tau v) - V(x(\tau))}{\tau}. \tag{3.18}$$

Since V is locally Lipschitz, by (3.17) one easily checks that the second term in the righthand side of the above equation goes to 0 as $\tau \to 0$. To estimate the first term in the righthand side, we first observe that

$$\begin{aligned} V(x(\tau)) &= \sup_{y(\cdot)\in\mathcal{S}_{x(\tau)}} \int_0^{+\infty} e^t \alpha(y(t))\, dt \\ &= \sup_{y(\cdot)\in\mathcal{S}_{x(\tau)}} e^{-\tau} \int_0^{+\infty} e^{t+\tau} \alpha(y(t))\, dt. \end{aligned}$$

For any $y = y(\cdot) \in \mathcal{S}_{x(\tau)}$, define

$$x_y(t) = \begin{cases} x(t), & t \in [0,\tau]; \\ y(t-\tau), & t > \tau. \end{cases}$$

Then $x_y(\cdot) \in \mathcal{S}_x$. We can rewrite $V(x(\tau))$ as

$$\begin{aligned} V(x(\tau)) &= \sup_{y(\cdot)\in\mathcal{S}_{x(\tau)}} e^{-\tau} \int_0^{+\infty} e^{t+\tau} \alpha(x_y(t+\tau))\, dt \\ &= \sup_{y(\cdot)\in\mathcal{S}_{x(\tau)}} e^{-\tau} \int_\tau^{+\infty} e^{t}\, \alpha(x_y(t))\, dt, \end{aligned}$$

from which it can be easily seen that

$$V(x(\tau)) \quad \le \sup_{x(\cdot)\in\mathcal{S}_x} e^{-\tau} \int_\tau^{+\infty} e^t\, \alpha(x(t))\, dt \le e^{-\tau} V(x),$$

which implies

$$\limsup_{\tau\to 0^+} \frac{V(x(\tau)) - V(x)}{\tau} \le -V(x).$$

By (3.18) one immediately deduces that

$$D_v^+ V(x) \le -V(x).$$

The proof of the lemma is complete. $\square$

Proof of Proposition 3.3. Take a decreasing sequence of numbers $\delta_n \downarrow 0$ with

$$\overline{\mathrm{B}}(\mathscr{A}, 2\delta_0) \subset \mathcal{O}.$$

Then for each δ_n one can find a $\mathcal{K}^\infty$ function $V_n \in C(\mathcal{O})$ satisfying all the properties in Lemma 3.4 with V and δ therein replaced by V_n and δ_n, respectively. In particular, V_n vanishes on $\mathscr{A}$ and satisfies:

$$V_n(x) > 0, \qquad \text{if } x \notin \overline{\mathrm{B}}(\mathscr{A}, \delta_n).$$

For each n we pick an $R_n > 0$ so that

$$\overline{\mathrm{B}}(\mathscr{A}, 2\delta_0) \subset K_n := \{x \in \mathcal{O} |\ V_n(x) \le R_n\}.$$

Define

$$W_n(x) = \begin{cases} V_n(x), & x \in K_n; \\ R_n, & x \in X \setminus K_n. \end{cases} \tag{3.19}$$

Then W_n is globally Lipschitz on X. Denote by L_n the Lipschitz constant of W_n. Let

$$V(x) = V_0(x) + \sum_{n=1}^{\infty} \frac{1}{2^n R_n L_n} W_n(x), \qquad x \in \mathcal{O}.$$

One trivially checks that V is a locally Lipschitz $\mathcal{K}^\infty$ function on $\mathcal{O}$. Moreover, $V|_{\mathscr{A}} = 0$.

Now we evaluate $D_v^+ V(x)$ for any $x \in \mathcal{O}$ and $v \in F(x)$. Observe (by (3.9)) that

$$D_v^+ W_n(x) \le \begin{cases} -V_n(x), & \text{if } x \in \mathrm{int} K_n; \\ 0, & \text{otherwise .} \end{cases} \tag{3.20}$$

So if we take $w_n \in C(\mathcal{O})$ as:

$$w_n(x) = \min\left(V_n(x),\, d(x, K_n^c)\right), \qquad x \in \mathcal{O},$$

then by (3.20) one finds that

$$D_v^+ W_n(x) \le -w_n(x).$$

Note that $0 \le w_n(x) \le R_n$ for all $x \in \mathcal{O}$. Define

$$w(x) = V_0(x) + \sum_{n=1}^{\infty} \frac{1}{2^n R_n L_n} w_n(x), \qquad x \in \mathcal{O}.$$

Then $w \in C(\mathcal{O})$. We verify that

$$w(x) > 0, \qquad \forall\, x \in \mathcal{O} \setminus \mathscr{A}.$$

There are two possibilities.

(1) "$x \notin \overline{\mathrm{B}}(\mathscr{A}, \delta_0)$". In this case we directly have

$$w(x) \geq V_0(x) > 0.$$

(2) "$x \in \overline{\mathrm{B}}(\mathscr{A}, \delta_0)$". We pick an n large enough so that $x \notin \overline{\mathrm{B}}(\mathscr{A}, \delta_n)$. Then

$$x \in \overline{\mathrm{B}}(\mathscr{A}, \delta_0) \setminus \overline{\mathrm{B}}(\mathscr{A}, \delta_n) \subset \mathrm{int} K_n \setminus \overline{\mathrm{B}}(\mathscr{A}, \delta_n).$$

Since both $V_n(x)$ and $d(x, K_n^c)$ are positive, by the definition of w_n we deduce that $w_n(x) > 0$. Therefore

$$w(x) \geq \frac{1}{2^n R_n L_n} w_n(x) > 0.$$

Now it is easy to deduce that

$$D_v^+ V(x) \leq D_v^+ V_0(x) + \sum_{n=1}^{\infty} \frac{1}{2^n R_n L_n} D_v^+ W_n(x) \leq -w(x),$$

which also follows

$$V(x) > 0, \qquad \forall\, x \in \mathcal{O} \setminus \mathscr{A}.$$

The proof is complete. □

3.2 Locally Lipschitz F vs smooth Lyapunov function

Proposition 3.5 *Let $\mathcal{O}$ be an open attractor neighborhood of $\mathscr{A}$. Then there exists a $\mathcal{K}^\infty$ function $V \in C^\infty(\mathcal{O})$ which vanishes on $\mathscr{A}$, such that*

$$\max_{v \in F(x)} \nabla V(x) \cdot v \leq -w(x), \qquad \forall\, x \in \mathcal{O}, \tag{3.21}$$

where $w \in C(\mathcal{O})$ is a nonnegative function satisfying (3.5).

Proof. Following the procedure as in [13], Section 5 (with very minor modifications), one can obtain a Lyapunov function V of $\mathscr{A}$, which belongs to $C^\infty(\mathcal{O} \setminus \mathscr{A})$ and satisfies (3.21), by smoothing the Lipschitz one given in Proposition 3.3 above. Further applying the following lemma, which is a slightly modified version of Lemma 4.3 in [40], we immediately get a smooth Lyapunov function on $\mathcal{O}$, as desired. □

Lemma 3.6 *Assume that $V \in C(\mathcal{O}) \bigcap C^\infty(\mathcal{O} \setminus \mathscr{A})$, and that*

$$V|_{\mathscr{A}} = 0, \quad V|_{\mathcal{O} \setminus \mathscr{A}} > 0.$$

Then there exists a nonnegative function $\beta \in C^\infty([0, \infty))$ with

$$\beta^{(k)}(0) = 0 \ \ (k = 0, 1, \cdots), \qquad \lim_{t \to +\infty} \beta(t) = +\infty$$

and

$$\beta'(t) > 0, \qquad \forall\, t > 0,$$

such that $W(x) = \beta(V(x))$ *is a* C^∞ *function on* $\mathcal{O}$.

Proof. See Lemma 4.3 in [40]. $\Box$

3.3 The general case: proofs of Theorems 3.1 and 3.2

To prove Theorems 3.1 and 3.2, we need the following approximation lemma. Closely related results can also be found in Lasry-Robert [34] etc.

Lemma 3.7 *For any* $\delta, R > 0$, *one can find a Lipschitz continuous multifunctions* $F_L : X \mapsto \mathcal{C}(X)$ *such that*

$$F(x) \subset F_L(x) \subset \overline{\text{con}}\, F(x + \delta\overline{B}_1) + \delta\overline{B}_1, \qquad \forall\, x \in \overline{B}_R, \tag{3.22}$$

where $\mathcal{C}(X)$ *denotes the family of nonempty compact convex subsets of* X.

Proof. The proof is directly adapted from that for Theorem 9.2.1 in [4].

Let $\delta, R > 0$ be given arbitrary. Then for every $x \in X$, by upper semicontinuity of F there exists $0 < r_x < \delta$ such that

$$F(\overline{\mathrm{B}}(x, r_x)) \subset F(x) + \delta\overline{\mathrm{B}}_1.$$

The family of the balls $\mathrm{B}(x, r_x/4)$ ($x \in \overline{\mathrm{B}}_R$) cover $\overline{\mathrm{B}}_R$. So there exist a finite number of balls $\mathrm{B}(x_i, r_{x_i}/4)$ ($1 \le i \le n$) that cover $\overline{\mathrm{B}}_R$.

We rewrite r_{x_i} as r_i for simplicity and take a Lipschitz partition of unity $\{a_i(x)\}_{1\le i\le n}$ subordinated to this finite covering. That is, each $a_i : X \to [0, 1]$ is a Lipschitz function that vanishes outside $\mathrm{B}(x_i, r_i/4)$, and

$$\sum_{1\le i\le n} a_i(x) = 1, \qquad x \in \overline{\mathrm{B}}_R.$$

Define F_L as:

$$F_L(x) = \sum_{1\le i\le n} a_i(x)\, F\left(\overline{\mathrm{B}}(x_i, r_i/4)\right), \qquad x \in X.$$

Then F_L is Lipschitz.

Let $x \in \overline{\mathrm{B}}_R$, and let $I(x)$ be the set of all i such that $a_i(x) \ne 0$. Then we must have

$$x \in \mathrm{B}(x_i, r_i/4), \qquad \forall\, i \in I(x).$$

Therefore for all $i, j \in I(x)$,

$$|x_i - x_j| \le |x_i - x| + |x - x_j| \le r_i/2. \tag{3.23}$$

Fix a $k \in I(x)$ satisfying
$$r_k = \max_{i \in I(x)} r_i.$$
Then $|x - x_k| \le r_k/4$, and (3.23) implies
$$x_i \in \mathrm{B}(x_k, r_k/2), \qquad \forall\, i \in I(x).$$
Consequently for all $i \in I(x)$,
$$\begin{aligned} F(\overline{\mathrm{B}}(x_i, r_i/4)) \quad &\subset F(\overline{\mathrm{B}}(x_k, r_k)) \subset F(x_k) + \delta \overline{\mathrm{B}}_1 \\ &\subset F(\overline{\mathrm{B}}(x, r_k)) + \delta \overline{\mathrm{B}}_1 \subset F(\overline{\mathrm{B}}(x, \delta)) + \delta \overline{\mathrm{B}}_1 \subset \overline{\mathrm{con}}\, F(x + \delta \overline{\mathrm{B}}_1)) + \delta \overline{\mathrm{B}}_1. \end{aligned}$$
Since the last term of this relation is convex, from the definition of F_L we see that
$$F_L(x) \subset \overline{\mathrm{con}}\, F(x + \delta \overline{\mathrm{B}}_1)) + \delta \overline{\mathrm{B}}_1.$$

There remains to verify that $F(x) \subset F_L(x)$. Assume that $y \in F(x)$. Then
$$y \in F(x) \subset F(\overline{\mathrm{B}}(x_i, r_i/4)), \qquad \forall\, i \in I(x).$$
Therefore
$$y = \sum_{i \in I(x)} a_i(x) y \in \sum_{i \in I(x)} a_i(x) F(\overline{\mathrm{B}}(x_i, r_i/4)) = F_L(x).$$

The proof is complete. $\square$

Now we are in a position to prove Theorems 3.2 and 3.1.

Proof of Theorem 3.1. Let $\mathcal{O}$ be an open attractor neighborhood of $\mathscr{A}$. Take a sequence of compact neighborhoods K_n of $\mathscr{A}$ with
$$K_1 \subset \mathrm{int} K_2 \subset \cdots \subset K_n \subset \mathrm{int} K_{n+1} \cdots, \quad \mathcal{O} = \cup_{n \ge 1} K_n.$$
For each K_n, since $\mathscr{A}$ attracts K_n we deduce that $\mathcal{R}(t) K_n$ is bounded for $t \ge 0$. Hence there exists an $R_n > 0$ such that
$$\mathcal{R}(t) K_n \subset \mathrm{B}_{R_n}, \qquad t \ge 0.$$
Let $U_n = \mathcal{O} \cap \mathrm{B}_{2R_n}$. Then clearly
$$K_n \subset \Omega^{U_n}(\mathscr{A}) \subset \mathcal{O},$$
where $\Omega^{U_n}(\mathscr{A})$ is the attraction basin of $\mathscr{A}$ in U_n; see (2.1) for the definition. By virtue of Theorem 2.7 there exists $\delta > 0$ such that the inflated system (2.2) has an attractor $\mathscr{A}(\delta)$ with
$$\mathscr{A}(\delta) \subset \mathrm{int}\, K_n \cap \mathrm{B}(\mathscr{A}, 1/n), \quad K_n \subset \Omega^{U_n}(\mathscr{A}(\delta)).$$

Thanks to Lemma 3.7 we can take a Lipschitz function $F_L : X \mapsto \mathcal{C}(X)$ such that

$$F(x) \subset F_L(x) \subset \overline{\text{con}}\, F(x + \delta\overline{\mathrm{B}}_1) + \delta\overline{\mathrm{B}}_1, \qquad \forall\, x \in \overline{\mathrm{B}}_{3R_n}. \tag{3.24}$$

Consider the system

$$x'(t) \in F_L(x(t)). \tag{3.25}$$

The second inclusion in (3.24) implies that (3.25) has an attractor, denoted by $\mathscr{A}_n$. We infer from (3.24) that

$$\mathscr{A} \subset \mathscr{A}_n \subset \mathscr{A}(\delta).$$

We claim that

$$\Omega^{U_n}(\mathscr{A}_n) \subset \Omega^{U_n}(\mathscr{A}) \subset \mathcal{O}. \tag{3.26}$$

Indeed, since $K_n \subset \Omega^{U_n}(\mathscr{A}(\delta)) \cap \Omega^{U_n}(\mathscr{A})$, it is easy to see that

$$K_n \subset \Omega^{U_n}(\mathscr{A}_n).$$

Assume that $x \in \Omega^{U_n}(\mathscr{A}_n)$. Let $\mathcal{R}_{F_L}(t)$ be the multi-valued semiflow generated by (3.25). Then by $\mathscr{A}_n \subset \mathscr{A}(\delta) \subset \text{int}K_n$ we deduce that $\mathcal{R}_{F_L}(t)x \subset \text{int}K_n$ for t sufficiently large. It then follows that

$$\mathcal{R}(t)x \subset \mathcal{R}_{F_L}(t)x \subset \text{int}K_n \subset K_n \tag{3.27}$$

for t sufficiently large. Because $\mathscr{A}$ attracts K_n, we conclude by (3.27) that $\mathscr{A}$ attracts x. Note that $x \in \Omega^{U_n}(\mathscr{A}_n)$ also implies

$$\mathcal{R}(t)x \subset \mathcal{R}_{F_L}(t)x \subset U_n, \qquad \text{for all } t \geq 0.$$

Hence $x \in \Omega^{U_n}(\mathscr{A})$. This proves (3.26).

$\mathcal{O}_n := \Omega^{U_n}(\mathscr{A}_n)$ is an open attractor neighborhood of $\mathscr{A}_n$. Therefore Proposition 3.5 implies that there exists a $\mathcal{K}^\infty$ function $V_n \in C^\infty(\mathcal{O}_n)$ which vanishes on $\mathscr{A}_n$ such that

$$\max_{v \in F_L(x)} \nabla V_n(x) \cdot v \leq -w_n(x), \qquad \forall\, x \in \mathcal{O}_n, \tag{3.28}$$

where $w_n \in C(\mathcal{O}_n)$ is a nonnegative function satisfying:

$$w_n|_{\mathscr{A}_n} = 0, \quad w_n|_{\mathcal{O}_n \setminus \mathscr{A}_n} > 0. \tag{3.29}$$

For each n take an $a_n > 0$ sufficiently large so that

$$K_n \subset D_n := \{x \in \mathcal{O}_n |\ V_n(x) \leq a_n\} \subset \mathcal{O}.$$

Define

$$\psi_n(x) = \begin{cases} V_n(x), & x \in D_n; \\ a_n, & x \in X \setminus D_n. \end{cases}$$

Clearly $\psi_n \in C^\infty(X \setminus \partial D_n)$. We will make a slight modification to ψ_n to obtain a smooth function $\phi_n \in C^\infty(X)$. For this purpose we first note that ψ_n is globally Lipschitz on X. Therefore there exists a $C > 0$ such that

$$|\psi_n(y) - \psi_n(x)| \le C|x - y|, \qquad \forall\, x \in \partial D_n,\ y \in X. \tag{3.30}$$

Set

$$G(s) = \begin{cases} \operatorname{sgn}(s) e^{-1/s^2}, & s > 0; \\ 0, & s = 0, \end{cases}$$

where sgn$(\cdot)$ is the signal function. Then $G \in C^\infty(\mathbb{R}^1)$, and

$$G^{(n)}(0) = 0 \ (\forall\, n \ge 0), \qquad G'(s) > 0 \ (\,\forall\, x \ne 0\,).$$

Define

$$\phi_n(x) = G\left(\psi_n(x) - a_n\right) + G(a_n).$$

By (3.30) one easily checks that $\phi_n \in C^\infty(X)$ with

$$\frac{\partial^l \phi_n(x)}{\partial x_{i_1} \cdots \partial x_{i_l}} = 0, \qquad \forall\, x \in \partial D_n, \quad l = 1, 2, \cdots,$$

where $1 \le i_k \le m$. Clearly ϕ_n satisfies:

$$\phi_n|_{\mathscr{A}_n} = 0, \qquad \phi_n|_{\mathscr{A}_n^c} > 0, \qquad \phi_n|_{D_n^c} = G(a_n). \tag{3.31}$$

For $x \in \mathrm{int} D_n$ and $v \in F(x) \subset F_L$, we have

$$\begin{aligned} \nabla \phi_n(x) \cdot v &= G'\left(\psi_n(x) - a_n\right) \nabla \psi_n(x) \cdot v \\ &= G'\left(\psi_n(x) - a_n\right) \nabla V_n(x) \cdot v \le -G'\left(\psi_n(x) - a_n\right) w_n(x). \end{aligned} \tag{3.32}$$

Let

$$\widetilde{w}_n(x) = \begin{cases} G'\left(\psi_n(x) - a_n\right) w_n(x), & x \in D_n\,; \\ 0, & x \in D_n^c\,. \end{cases}$$

Since D_n is a compact subset of $\mathcal{O}_n$ and $w_n \in C(\mathcal{O}_n)$, one easily sees that $\widetilde{w}_n$ is continuous and bounded on X. (3.32) amounts to say that

$$\nabla \phi_n(x) \cdot v \le -\widetilde{w}_n(x), \qquad \forall\, x \in \mathrm{int} D_n,\ v \in F(x). \tag{3.33}$$

The above estimate naturally holds if $x \in X \setminus \mathrm{int} D_n$, as in this case both sides vanish.

We also note that

$$\widetilde{w}_n(x) > 0, \qquad \forall\, x \in \mathrm{int} D_n \setminus \mathscr{A}_n. \tag{3.34}$$

ϕ_n is constant on D_n^c, so $||\phi_n||_{C^k(X)} = ||\phi_n||_{C^k(D_n)}$ for all integers $k \geq 0$. Let

$$c_n = ||\phi_n||_{C^n(X)} + ||\widetilde{w}_n||_{C(X)} + 1.$$

Define

$$V(x) = \gamma \sum_{n=1}^{\infty} \frac{1}{2^n c_n} \phi_n(x), \qquad x \in X, \tag{3.35}$$

where $\gamma = 1/\sum_{n=1}^{\infty} \frac{1}{2^n c_n} G(a_n)$. For any $l \in \mathbb{N}$ and $1 \leq i_1, \cdots, i_l \leq m$, because

$$\sum_{n=l}^{\infty} \frac{1}{2^n c_n} \left| \frac{\partial^l \phi_n(x)}{\partial x_{i_1} \cdots \partial x_{i_l}} \right| \leq \sum_{n=l}^{\infty} \frac{1}{2^n c_n} ||\phi_n||_{C^l(X)} \leq \sum_{n=l}^{\infty} \frac{1}{2^n c_n} ||\phi_n||_{C^n(X)} \leq \sum_{n=l}^{\infty} \frac{1}{2^n},$$

we deduce that the series $\sum_{n=1}^{\infty} \frac{1}{2^n c_n} \frac{\partial^l \phi_n(x)}{\partial x_{i_1} \cdots \partial x_{i_l}}$ is uniformly convergent on X. It then follows that $V \in C^\infty(X)$. Moreover, for any $x \in X$ and $v \in F(x)$, we have

$$\nabla V(x) \cdot v \leq \gamma \sum_{n=1}^{\infty} \frac{1}{2^n c_n} \nabla \phi_n(x) \cdot v \leq -w(x), \tag{3.36}$$

where

$$w(x) = \gamma \sum_{n=1}^{\infty} \frac{1}{2^n c_n} \widetilde{w}_n(x).$$

There remains to check that V and w satisfies all the other properties required in the Theorem.

Let $x \in \mathcal{O}^c$. Then $x \in D_n^c$ for all $n \in \mathbb{N}$. By the construction of V and w, we have

$$V(x) = \gamma \sum_{n=1}^{\infty} \frac{1}{2^n c_n} \phi_n(x) = \gamma \sum_{n=1}^{\infty} \frac{1}{2^n c_n} G(a_n) = 1,$$

$$w(x) = \gamma \sum_{n=1}^{\infty} \frac{1}{2^n c_n} \widetilde{w}_n(x) = 0.$$

Assume that $x \in \mathcal{O} \setminus \mathscr{A}$. In this case one can find a $j \in \mathbb{N}$ sufficiently large so that

$$x \in K_{j-1} \setminus \mathscr{A}_j \subset \mathrm{int} K_j \setminus \mathscr{A}_j \subset \mathrm{int} D_j \setminus \mathscr{A}_j.$$

(Recall that $\mathcal{O} = \cup_{n \geq 1} K_n$ and $\mathscr{A}_n \subset \mathrm{B}(\mathscr{A}, 1/n)$.) (3.34) then implies that

$$w(x) = \gamma \sum_{n=1}^{\infty} \frac{1}{2^n c_n} \widetilde{w}_n(x) \geq \gamma \frac{1}{2^j c_j} \widetilde{w}_j(x) > 0. \tag{3.37}$$

Finally it is trivial to examine that both V and w vanish on $\mathscr{A}$. That V is positive on $\Omega \setminus \mathscr{A}$ directly follows from (3.36) and (3.37).

The proof of the lemma is complete. $\square$

Proof of Theorem 3.2. Let V be a Lyapunov function of $\mathscr{A}$ given by Theorem 3.1. Define

$$L(x) = \eta(V(x)), \qquad x \in \mathcal{O},$$

where $\eta(s) = -\ln(1-s)$ $(s \in [0,1))$. Then L is a $\mathcal{K}^\infty$ Lyapunov function of $\mathscr{A}$ on $\mathcal{O}$ that satisfies all the desired properties in the theorem. □

4 Smooth Morse-Lyapunov Functions

Let $\mathscr{A}$ be an attractor of (1.1) with Morse decomposition $\mathcal{M} = \{M_1, \cdots, M_l\}$, and let

$$\mathcal{D} = \cup_{1 \le k \le l} M_k.$$

A continuous function V on the attraction basin $\Omega = \Omega(\mathscr{A})$ is said to be a *Morse-Lyapunov function* (M-L function in short) of $\mathcal{M}$ on Ω, if

(1) V is constant on each Morse set M_k;

(2) $V(x(t))$ is strictly decreasing in t for any solution $x(\cdot)$ of (1.1) in $\Omega \setminus \mathcal{D}$.

An M-L function V of $\mathcal{M}$ is said to be a *strict M-L function*, if in addition it satisfies:

$$V(M_i) < V(M_j)$$

whenever $i < j$ with $M_i \ne \emptyset \ne M_j$.

The main result in this section is contained in the following theorem.

Theorem 4.1 (**Existence of smooth M-L functions**) *$\mathcal{M}$ has a radially unbounded strict M-L function $V \in C^\infty(\Omega)$ such that*

$$\max_{v \in F(x)} \nabla V(x) \cdot v \le -w(x), \qquad \forall\, x \in \Omega, \tag{4.1}$$

where $w \in C(\Omega)$ is a nonnegative function satisfying

$$w|_{\mathcal{D}} = 0, \quad w|_{\Omega \setminus \mathcal{D}} > 0. \tag{4.2}$$

Proof. Let

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_l = \mathscr{A}$$

be the Morse filtration of $\mathcal{M}$.

For $k = l$, by Theorem 3.2 there exists a $\mathcal{K}^\infty$ function $V_l \in C^\infty(\Omega)$ such that

$$V_l|_{A_l} = V_l|_{\mathscr{A}} = 0,$$

$$\max_{v \in F(x)} \nabla V_l(x) \cdot v \le -w_l(x), \qquad \forall\, x \in \Omega,$$

where $w_l \in C(\Omega)$ is a nonnegative function satisfying

$$w_l|_{\mathscr{A}_l} = 0, \quad w_l|_{\Omega \setminus \mathscr{A}_l} > 0.$$

For each $1 \le k \le l-1$, it follows from Theorem 3.1 that there exists a $V_k \in C^\infty(X)$ such that

$$V_k|_{A_k} = 0, \quad V_k|_{\Omega(A_k)^c} = 1,$$

and

$$\max_{v \in F(x)} \nabla V_k(x) \cdot v \le -w_k(x), \qquad \forall\, x \in X,$$

where $w_k \in C(X)$ is a nonnegative function with

$$w_k|_{A_k \cup \Omega(A_k)^c} = 0, \quad w_k|_{\Omega(A_k) \setminus A_k} > 0.$$

Define $V \in C(\Omega)$ as:

$$V(x) = \sum_{1 \le k \le l} V_k(x), \qquad x \in \Omega.$$

We show that V has all the required properties with

$$w(x) = \sum_{1 \le k \le l} w_k(x).$$

Let $x \in \Omega$, and $v \in F(x)$. Then

$$\nabla V(x) \cdot v \le \sum_{1 \le k \le l} \nabla V_k(x) \cdot v \le -\sum_{1 \le k \le l} w_k(x) = -w(x).$$

This verifies (4.1). In what follows we examine the validity of (4.2).

Let $x \in \mathcal{D}$. We may assume that $x \in M_k = A_k \cap A_{k-1}^*$ for some k. If $i \ge k$, then we have $x \in A_i$, and hence $w_i(x) = 0$. On the other hand, in case $i \le k-1$ we deduce by $x \in A_{k-1}^*$ that

$$x \in \Omega(A_{k-1})^c \subset \Omega(A_i)^c,$$

which also implies $w_i(x) = 0$. In conclusion,

$$w_i(x) = 0, \qquad \text{for all } 1 \le i \le l.$$

Consequently $w(x) = 0$.

Now assume $x \in \Omega \setminus \mathcal{D}$. Then there is a smallest k such that $x \in \Omega(A_k)$. We claim that $x \notin A_k$. Indeed, if $x \in A_k$, then since (A_{k-1}, M_k) is an attractor-repeller pair in A_k, we have either $x \in M_k$, or $x \in \Omega(A_{k-1})$. The former case contradicts to that $x \in \Omega \setminus \mathcal{D}$, and the latter one to the definition of k. Hence the claim holds true. Now that $x \in \Omega(A_k) \setminus A_k$, one finds that $w_k(x) > 0$. Thus we have

$$w(x) \ge w_k(x) > 0.$$

Finally we check that V is a strict M-L function of $\mathscr{A}$. Recall that

$$M_k = A_k \cap A_{k-1}^* = A_k \cap (\mathscr{A} \setminus \Omega(A_{k-1})) = A_k \cap \Omega(A_{k-1})^c.$$

Thus if $i \leq k-1$, then $M_k \subset \Omega(A_{k-1})^c \subset \Omega(A_i)^c$, which implies, in case $M_k \neq \emptyset$, that

$$V_i(M_k) = 1, \qquad \text{for } 1 \leq i \leq k-1.$$

On the other hand if $i \geq k$, then $M_k \subset A_k \subset A_i$. Therefore

$$V_i(M_k) = 0, \qquad \text{for } i \geq k.$$

Hence if M_k is nonvoid, then we conclude that

$$V(M_k) = \sum_{1 \leq i \leq l} V_i(M_k) = k-1.$$

The proof is complete. $\square$

5 Morse Theory

In this section we try to establish a Morse theory for attractors. First, we prove that all the open attractor neighborhoods of an attractor have the same homotopy type by employing smooth Lyapunov functions. Then, based on this fact we introduce the concept of critical groups for Morse sets and establish Morse equations and inequalities.

For any $a \in \mathbb{R}^1$, we will denote by V_a the level set of a function $V : R^m \to \mathbb{R}^1$,

$$V_a = \{x \in \mathbb{R}^m \mid V(x) \leq a\}.$$

5.1 Homotopy equivalence of open attractor neighborhoods

Let $\mathscr{A}$ be an attractor of the system (1.1). We shall prove that all the open neighborhoods of $\mathscr{A}$ have the same homotopy type.

The following lemma will play an important role.

Lemma 5.1 *Let $\mathcal{O}_i$ $(i=1,2)$ be two open attractor neighborhoods of $\mathscr{A}$, and let V_i $(i=1,2)$ be Lyapunov functions of $\mathscr{A}$ on $\mathcal{O}_i$ given by Theorem 3.2, respectively.*

Then for any compact set $K \subset (\mathcal{O}_1 \cap \mathcal{O}_2) \setminus \mathscr{A}$, there exists a smooth vector field Ψ defined on X such that

$$\nabla V_i(x) \cdot \Psi(x) < 0, \qquad \forall\, x \in K, \;\; i = 1, 2.$$

Proof. Clearly $\mathcal{O} := (\mathcal{O}_1 \cap \mathcal{O}_2) \setminus \mathscr{A}$ is open. For each $x \in \mathcal{O}$ we fix a $v_x \in F(x)$. Then

$$\nabla V_i(x) \cdot v_x < 0, \qquad i = 1, 2.$$

Take an $r_x > 0$ sufficiently small so that $\mathrm{B}(x, r_x) \subset \mathcal{O}$, and

$$\nabla V_i(y) \cdot v_x < 0, \qquad \forall\, y \in \mathrm{B}(x, r_x), \;\; i = 1, 2.$$

Then the family of balls $\{\mathrm{B}(x, r_x)\}_{x \in K}$ forms an open covering of K, therefore by compactness of K it has a finite subcovering $\mathcal{V} = \{\mathrm{B}(x_k, r_{x_k})\}_{1 \le k \le n}$. Let $a_k \in C^\infty(X)$ $(1 \le k \le n)$ be a smooth unit partition of K subordinated to $\mathcal{V}$, namely, each a_k vanishes outside $\mathrm{B}(x_k, r_{x_k})$, and

$$\sum_{1 \le k \le n} a_k(x) \equiv 1, \qquad x \in K.$$

Define Ψ on X as:

$$\Psi(x) = \sum_{1 \le k \le n} a_k(x) v_{x_k} \,, \qquad x \in X.$$

Then $\Psi \in C^\infty$. For each $x \in K$, we have

$$\begin{aligned} \nabla V_i(x) \cdot \Psi(x) \;\; &= \textstyle\sum_{1 \le k \le n} a_k(x) \; (\nabla V_i(x) \cdot v_{x_k}) \\ &\le \max_{1 \le k \le n} (\nabla V_i(x) \cdot v_{x_k}) \textstyle\sum_{1 \le k \le n} a_k(x) \\ &= \max_{1 \le k \le n} (\nabla V_i(x) \cdot v_{x_k}) < 0. \end{aligned}$$

This finishes the proof of the lemma. $\square$

Proposition 5.2 *Let W, W' be two open attractor neighborhoods of $\mathscr{A}$. Then there exists a compact attractor neighborhood $\mathcal{O}$ of $\mathscr{A}$ such that $\mathcal{O}$ is a strong deformation retract of both W and W'.*

Consequently, all open attractor neighborhoods of $\mathscr{A}$ have the same homotopy type.

Proof. Let V, V' be smooth Lyapunov functions of $\mathscr{A}$ on W and W' given by Theorem 3.2, respectively. Take two positive numbers $0 < \delta < \varepsilon$ sufficiently small such that

$$V'_\delta \subset V_{\varepsilon/2} \subset V_\varepsilon \subset W'.$$

We first show that V_ε and V'_δ are strong deformation retracts of W and W', respectively.

Indeed, let $S(t)$ be the semiflow on the phase space $X := W$ generated by the system:

$$x'(t) = -\nabla V(x(t)), \qquad x(t) \in W. \tag{5.1}$$

Namely, $S(t)x$ is the unique solution of the the system for each $x \in W$. It is clear that $S(t)$ is well defined on W. Moreover, since $\nabla V(x) \ne 0$ outside $\mathscr{A}$, one easily deduces that $S(t)$ has a global attractor $\mathcal{A} \subset \mathscr{A}$.

Define a function $t(x)$ on W as:

$$t(x) = \begin{cases} \sup\{t \geq 0 |\ S([0,t))x \subset W \setminus V_\varepsilon\}, & x \in W \setminus V_\varepsilon; \\ 0, & x \in V_\varepsilon. \end{cases}$$

As V_ε is a neighborhood of $\mathcal{A}$ and $\mathcal{A}$ attracts x, we see that $t(x)$ is finite for each $x \in W$. Because ∇V and ∂V_ε is transversal at any point $x \in \partial V_\varepsilon$, by the basic knowledge on geometric theory of ODEs we know that $t(x)$ is continuous in x.

Define

$$H(\sigma, x) = S(\sigma\, t(x))x, \qquad x \in W.$$

Then $H : [0,1] \times W \to W$ is continuous and satisfies:

$$H(0, \cdot) = \mathrm{id}_W, \qquad H(1, W) \subset V_\varepsilon,$$

$$H(\sigma, x) = x, \qquad \forall\, (\sigma, x) \in [0,1] \times V_\varepsilon.$$

That is, V_ε is a strong deformation retract of W.

The same argument applies to show that V'_δ is a strong deformation retract W'.

We claim that V'_δ is a strong deformation retract of V_ε. For this purpose let $K = V_\varepsilon \setminus \mathrm{int} V'_\delta$. Then $K \subset (W \cap W') \setminus \mathscr{A}$ and is compact. It follows by Lemma 5.1 that there exists a smooth vector field Ψ defined on X such that

$$\nabla V(x) \cdot \Psi(x) < 0, \quad \nabla V'(x) \cdot \Psi(x) < 0 \tag{5.2}$$

for all $x \in K$. Consider the semiflow $T(t)$ generated by

$$x'(t) = \Psi(x(t)).$$

(5.2) implies that both V_ε and V'_δ are positively invariant with respect to $T(t)$; moreover, the vector field Ψ is transversal to both ∂V_ε and $\partial V'_\delta$. Making use of $T(t)$ one can easily construct a continuous function $G : [0,1] \times V_\varepsilon \to V_\varepsilon$, which is a strong deformation from V_ε to V'_δ and thus proves our claim. Since the argument is quite similar as above, we omit the details.

Let $\mathcal{O} = V'_\delta$. Recall that $\mathcal{O}$ is a strong deformation retract of W'. In what follows we show that $\mathcal{O}$ is also a strong deformation retract of W, thus finishes the proof of the proposition. For this purpose we define $\Theta : [0,1] \times W \to W$ as follows:

$$\Theta(\sigma, x) = \begin{cases} H(2\sigma, x), & 0 \leq \sigma \leq 1/2, \ \ x \in W; \\ G\left(2\sigma - 1, H(1, x)\right), & 1/2 \leq \sigma \leq 1, \ \ x \in W. \end{cases}$$

Clearly Θ is continuous, and $\Theta(0, \cdot) = \mathrm{id}_W$. We observe that

$$\Theta(1, W) = G(1, H(1, W)) \subset G(1, V_\varepsilon) \subset \mathcal{O}.$$

Let $x \in \mathcal{O}$ ($\subset V_\varepsilon$). Then

$$\Theta(\sigma, x) = H(2\sigma, x) = x, \qquad \text{if } \sigma \le 1/2,$$

and

$$\Theta(\sigma, x) = G(2\sigma - 1,\ H(1, x)) = G(2\sigma - 1, x) = x, \qquad \text{if } \sigma > 1/2,$$

Therefore Θ is a strong deformation from W to $\mathcal{O}$. □

5.2 Critical groups of Morse sets

We denote by H_* the usual singular homology theory with coefficients in a given Abelian group $\mathscr{G}$. Let $\mathscr{A}$ be an attractor of (1.1) with the attraction basin $\Omega = \Omega(\mathscr{A})$, and let $\mathcal{M} = \{M_1, \cdots, M_l\}$ be a Morse decomposition of $\mathscr{A}$ with Morse filtration

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_l = \mathscr{A}.$$

We first prove the following basic fact.

Proposition 5.3 *Let W and W' be two open attractor neighborhoods of A_k, and let U and U' be two open attractor neighborhoods of A_{k-1}. Then*

$$H_*(W, U) \cong H_*(W', U').$$

Proof. Proposition 5.2 allows us to pick compact attractor neighborhoods $\mathcal{O}$ of A_k and K of A_{k-1} with $K \subset \mathcal{O}$, such that $\mathcal{O}$ (resp. K) is a strong deformation retract of both W and W' (resp. U and U'). Consider the commutative diagram:

$$\begin{array}{ccccccccc}
H_q(K) & \xrightarrow{i_*} & H_q(\mathcal{O}) & \xrightarrow{j_*} & H_q(\mathcal{O}, K) & \xrightarrow{\partial} & H_{q-1}(K) & \xrightarrow{i_*} & H_{q-1}(\mathcal{O}) \\
\downarrow i_* & & \downarrow i_* & & \downarrow i_* & & \downarrow i_* & & \downarrow i_* \\
H_q(U) & \xrightarrow{i_*} & H_q(W) & \xrightarrow{j_*} & H_q(W, U) & \xrightarrow{\partial} & H_{q-1}(U) & \xrightarrow{i_*} & H_{q-1}(W)
\end{array}$$

The upper and lower rows present the exact homology sequences for the pairs $(\mathcal{O}, K)$ and (W, U), respectively. The homomorphisms i_*'s in the vertical arrows are induced by inclusions. Since the vertical arrows number 1,2,4 and 5 are isomorphisms, we immediately conclude by the well known "Five-lemma" (see [55], Lemma IV.5.11) that

$$H_q(W, U) \cong H_q(\mathcal{O}, K).$$

Similarly we also have

$$H_q(W', U') \cong H_q(\mathcal{O}, K).$$

This completes the proof of the proposition. □

Now let us introduce the concept of *critical group* for Morse sets.

Definition 5.4 *The critical group $C_*(M_k)$ of Morse set M_k is defined to be the homology theory given by*

$$C_q(M_k) = H_q(W, U), \qquad q = 0, 1, \cdots,$$

where W and U are open attractor neighborhoods of A_k and A_{k-1}, respectively.

Remark 5.5 *If $M_k = \emptyset$, then we necessarily have $A_k = A_{k-1}$. In such a case both W and U are open attractor neighborhoods of A_{k-1}. Take a $K \subset W \cap U$ so that K is a strong deformation retract of both W and U. Then one finds that*

$$C_*(M_k) = H_*(W, U) \cong H_*(K, K) = 0.$$

Proposition 5.6 *Let V be a C^1 strict M-L function of $\mathcal{M}$ satisfying (4.1) and (4.2). Take two real numbers $a < b$ such that*

(1) *M_k is the unique Morse set contained in $V^{-1}([a, b])$;*

(2) *if $M_k \neq \emptyset$, then $a < c_k < b$, where $c_k = V(M_k)$.*

Then

$$C_*(M_k) = H_*(V_b, V_a).$$

Proof. Choose a number $\varepsilon > 0$ small enough so that $a + \varepsilon < b - \varepsilon$. If $M_k \neq \emptyset$, we also require $a + \varepsilon < c_k < b - \varepsilon$. Using the semiflow $S(t)$ generated by the gradient system:

$$x'(t) = -\nabla V(x(t)),$$

it can be shown that $V_{b-\varepsilon}$ is a strong deformation retract of both $\text{int}V_b$ and V_b, and V_a is a strong deformation retract of $\text{int}V_{a+\varepsilon}$.

Note that $\text{int}V_b$ and $\text{int}V_{a+\varepsilon}$ are open attractor neighborhoods of A_k and A_{k-1}, respectively. Hence by the definition of critical groups we have

$$C_*(M_k) = H_*\left(\text{int}V_b, \text{int}V_{a+\varepsilon}\right).$$

On the other hand, using the same argument as in the proof of Proposition 5.3 one easily verifies

$$H_*\left(\text{int}V_b, \text{int}V_{a+\varepsilon}\right) \cong H_*\left(V_{b-\varepsilon}, V_a\right) \cong H_*\left(V_b, V_a\right),$$

and the conclusion of the proposition follows. □

Remark 5.7 *Let V, a, b and M_k be the same as in Proposition 5.6. If V has no critical points in M_k (and hence in $V^{-1}([a, b])$), then it can be easily shown that V_a is a strong deformation retract of V_b. It follows that $C_*(M_k) = 0$. As a matter of fact, we deduce in general that either $C_*(M_k) = 0$, or M_k contains at least a critical point of V.*

A particular but important case is that M_k consists of exactly one equilibrium z of the system (1.1). In this case if $C_*(z) \neq 0,$, then z is necessarily a critical point of V, i.e, $\nabla V(z) = 0$. By the basic knowledge in the theory of variational methods we conclude that

$$C_*(z) = H_*(V_b, V_a) \cong H_*(V_c, V_c \setminus \{z\}), \tag{5.3}$$

where $c = c_k = V(z)$; see, for instance, Chang [10]. Further by excision of homologies one deduces that for any neighborhood U of z, it holds that

$$C_*(z) \cong H_*(V_c, V_c \setminus \{z\}) \cong H_*\left(V_c \cap U,\, V_c \setminus \{z\} \cap U\right). \tag{5.4}$$

Note that if z is asymptotically stable, then (5.3) and (5.4) imply

$$C_*(z) \cong H_*(\{z\}) = \begin{cases} \mathscr{G}, & q = 0; \\ 0, & q \geq 1, \end{cases} \tag{5.5}$$

Example 6.1. Consider the following differential inclusion which relates to the generalized equations governing Chua's circuit [7]:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \in A \begin{pmatrix} x_1 - k\,\mathrm{Sgn}(x_1) \\ x_2 \\ x_3 + k\,\mathrm{Sgn}\ (x_1) \end{pmatrix}, \quad A = \begin{pmatrix} -\alpha(b+1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}, \tag{5.6}$$

where $\mathrm{Sgn}(x)$ corresponds to the signal function,

$$\mathrm{Sgn}(x) = 1\ (x > 0), \quad \mathrm{Sgn}(x) = -1\ (x < 0), \quad \mathrm{Sgn}(0) = [-1, 1].$$

Taking $\alpha = -1$, $\beta = 288$, $b = -36$, and $k = 1$, the system reads:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \in A_0 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 35\mathrm{Sgn}(x_1) \\ 0 \\ 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -35 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & -288 & 0 \end{pmatrix}. \tag{5.7}$$

Simple computations show that all the eigenvalues of A_0 are negative, so the system (5.6) is dissipative and has a global attractor $\mathscr{A}$. (5.6) has three equilibria:

$$E_1 = (-1, 0, 1), \quad E_2 = (1, 0, -1), \quad E_3 = (0, 0, 0),$$

where E_1 and E_2 are asymptotically stable (therefore each one is an attractor). Let

$$A_0 = \emptyset, \quad A_1 = \{E_1\}, \quad A_2 = \{E_1, E_2\}, \quad A_3 = \mathscr{A}.$$

Then $\{A_k\}$ is an increasing attractor sequence which yields a Morse decomposition $\mathcal{M} = \{M_1, M_2, M_3\}$ of $\mathscr{A}$ with

$$M_1 = \{E_1\}, \quad M_2 = \{E_2\}, \quad E_3 \in M_3.$$

For simplicity we take the coefficients group $\mathcal{G} = \mathbb{Z}$. By (5.5) one finds that

$$C_q(M_i) = \begin{cases} \mathbb{Z}, & q = 0; \\ 0, & q \geq 1, \end{cases} \qquad i = 1, 2.$$

Now let us compute $C_*(M_3)$. Choose open attractor neighborhoods U_1 of E_1 and U_2 of E_2 with $U_1 \cap U_2 = \emptyset$. Then $U = U_1 \cup U_2$ is an open attractor neighborhood of A_2. We observe that

$$H_*(U_i) = C_*(U_i) = H_*(\{E_i\}).$$

Hence

$$H_q(U) \cong H_q(U_1) \oplus H_q(U_2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & q = 0; \\ 0, & q \geq 1. \end{cases}$$

Let $W = \mathbb{R}^3$. By definition of critical group we have

$$C_*(M_3) = H_*(W, U).$$

Using the exact sequence

$$\cdots \longrightarrow H_1(W, U) \xrightarrow{\partial} H_0(U) \xrightarrow{i_*} H_0(W) \xrightarrow{j_*} H_0(W, U) \longrightarrow 0,$$

one finds that

$$H_0(W, U) \cong H_0(W)/\mathrm{Ker}\,(j_*) = H_0(W)/\mathrm{Im}\,(i_*).$$

It is easy to see that $\mathrm{Im}\,(i_*) = H_0(W)$. Thus we obtain $H_0(W, U) = 0$.

To compute $H_1(W, U)$, we consider the long exact sequence of reduced homologies

$$\cdots \longrightarrow \widetilde{H}_q\,(W) \xrightarrow{j_*} \widetilde{H}_q\,(W, U) \xrightarrow{\partial} \widetilde{H}_{q-1}\,(U) \xrightarrow{i_*} \widetilde{H}_{q-1}\,(W) \longrightarrow \cdots.$$

Noticing that $\widetilde{H}_q\,(W) = \widetilde{H}_{q-1}\,(W) = 0$ for all $q \geq 0$, one deduces that

$$\widetilde{H}_q\,(W, U) \cong \widetilde{H}_{q-1}\,(U) = \begin{cases} \mathbb{Z}, & q = 1; \\ 0, & q > 1. \end{cases}$$

Hence by definition of reduced homologies we conclude that

$$H_q(W, U) = \widetilde{H}_q\,(W, U) = \begin{cases} \mathbb{Z}, & q = 1; \\ 0, & q > 1. \end{cases}$$

Therefore

$$C_q(M_3) = \begin{cases} \mathbb{Z}, & q = 1; \\ 0, & q \neq 1. \end{cases}$$

5.3 Morse inequalities and Morse equation

Now let us establish Morse inequalities and Morse equations for attractors. Let

$$\mathfrak{m}_q = \sum_{k=1}^{l} \operatorname{rank} C_q\left(M_k\right), \qquad q = 0, 1, \cdots . \tag{5.8}$$

$\mathfrak{m}_q$ is called the $q-$th **Morse type number** of $\mathcal{M}$.

Theorem 5.8 (**Morse inequality and equation**) *Let $\beta_q = \beta_q(\Omega) := \operatorname{rank} H_q(\Omega)$ be the q-th Betti number of the attraction basin Ω. Then the following inequalities and equation hold:*

$$\mathfrak{m}_0 \geq \beta_0,$$

$$\mathfrak{m}_1 - \mathfrak{m}_0 \geq \beta_1 - \beta_0,$$

$$\cdots\cdots$$

$$\mathfrak{m}_m - \mathfrak{m}_{m-1} + \cdots + (-1)^m \mathfrak{m}_0 = \beta_m - \beta_{m-1} + \cdots + (-1)^m \beta_0.$$

Remark 5.9 *If we define formal Poincaré-polynomials*

$$P_{\mathscr{A}}(t) = \sum_{q=0}^{m} \beta_q t^q, \qquad M_{\mathscr{A}}(t) = \sum_{q=0}^{m} \mathfrak{m}_q t^q,$$

then the Morse inequalities and Morse equation in above can be reformulated in a very simplified manner:

$$M_{\mathscr{A}}(t) - P_{\mathscr{A}}(t) = (1+t) Q_{\mathscr{A}}(t), \tag{5.9}$$

where $Q_{\mathscr{A}}(t) = \sum_{q=0}^{m} \gamma_q \, t^q$ is a a formal polynomial with γ_q being nonnegative integers.

To prove Theorem 5.8, we first need to recall some basic facts.

A real function Φ defined on a suitable family $D(\Phi)$ of pairs of spaces is said to be *subadditive*, if $W \subset Z \subset Y$ implies

$$\Phi(Y, W) \leq \Phi(Y, Z) + \Phi(Z, W).$$

If Φ is subadditive, then for any $Y_0 \subset Y_1 \subset \cdots \subset Y_n$ with $(Y_k, Y_{k-1}) \in D(\Phi)$,

$$\Phi(Y_n, Y_0) \leq \sum_{k=1}^{n} \Phi(Y_k, Y_{k-1}).$$

For any pair (Y, Z) of spaces, set

$$R_q(Y, Z) = \operatorname{rank} H_q(Y, Z) \quad (q\text{-th } \textit{Betti number}).$$

Define

$$\Phi_q(Y, Z) = \sum_{j=0}^{q} (-1)^{q-j} R_j(Y, Z), \qquad \chi(Y, Z) = \sum_{q=0}^{\infty} (-1)^q R_q(Y, Z).$$

$\chi(Y, Z)$ is usually called the *Euler number* of (Y, Z).

Lemma 5.10 [10, 45] *The functions R_q, Φ_q are subadditive, and χ are additive.*

Proof of Theorem 5.8. We may assume that all the Morse sets are nonvoid, as the critical group of such a Morse set is trivial. The following argument is quite standard as in the case of the classical Morse theory.

Let V be a C^1 strict M-L functionof $\mathcal{M}$, and let

$$c_k = V(M_k), \qquad 1 \le k \le l.$$

Take $a, b \in \mathbb{R}^1$ be such that

$$a < c_1 < c_2 < \cdots < c_l < b.$$

As c_1 is the minimum of V on Ω, we have

$$\emptyset = V_a \subset \mathscr{A} \subset V_b.$$

Taking $a_k \in \mathbb{R}^1$ $(k = 0, 1, \cdots, l)$ be such that

$$a = a_0 < c_1 < a_1 < c_2 < a_2 < \cdots < c_l < a_l = b,$$

by Lemma 5.10 one immediately deduces that

$$\sum_{i=1}^{l} \sum_{j=0}^{q} (-1)^{q-j} R_j \left(V_{a_i}, V_{a_{i-1}}\right) \ge \sum_{j=0}^{q} (-1)^{q-j} R_j \left(V_{a_l}, V_{a_0}\right),$$

that is,

$$\sum_{j=0}^{q} (-1)^{q-j} \mathfrak{m}_j \ge \sum_{j=0}^{q} (-1)^{q-j} R_j(V_b). \tag{5.10}$$

Noting that V has no critical point in $V^{-1}([b, +\infty))$, making use of the semiflow of the gradient system

$$x'(t) = -\nabla V(x(t)), \qquad x(t) \in \Omega,$$

it can be easily shown that V_b is a strong deformation retract of Ω. Therefore $H_*(V_b) = H_*(\Omega)$, and hence $R_j(V_b) = \beta_q$. This and (5.10) justify the Morse inequalities.

To prove the Morse equation, we observe that

$$\chi(V_b, V_a) = \sum_{q=0}^{m} (-1)^q R_q(V_b, V_a) = \sum_{q=0}^{m} (-1)^q \beta_q.$$

The additivity of χ also yields that

$$\begin{aligned} \chi(V_b, V_a) &= \textstyle\sum_{i=1}^{l} \chi \left(V_{a_i}, V_{a_{i-1}}\right) \\ &= \textstyle\sum_{i=1}^{l} \sum_{q=0}^{m} (-1)^q R_q \left(V_{a_i}, V_{a_{i-1}}\right) \\ &= \textstyle\sum_{q=0}^{m} (-1)^q \sum_{i=1}^{l} R_q \left(V_{a_i}, V_{a_{i-1}}\right) = \sum_{q=0}^{m} (-1)^q \mathfrak{m}_q. \end{aligned}$$

Therefore

$$\sum_{q=0}^{m}(-1)^q\beta_q = \sum_{q=0}^{m}(-1)^q\mathfrak{m}_q.$$

This is precisely what we desired. □

Remark 5.11 *If $\mathscr{A}$ is the global attractor of the flow, then β_q is the q-th Betti number of the phase space $X = \mathbb{R}^m$. Since X is contractable, we have*

$$H_q(X) = \begin{cases} \mathcal{G}, & q = 0; \\ 0, & q \neq 0. \end{cases}$$

Taking $\mathcal{G} = \mathbb{Z}$, one obtains

$$\beta_0 = 1, \quad \beta_q = 0 \ (q > 0).$$

Consequently the Morse inequalities and Morse equation read:

$$\mathfrak{m}_0 \geq 1,$$

$$\mathfrak{m}_1 - \mathfrak{m}_0 \geq -1,$$

$$\cdots\cdots$$

$$\mathfrak{m}_m - \mathfrak{m}_{m-1} + \cdots + (-1)^m\mathfrak{m}_0 = (-1)^m.$$

Example 6.2. Consider the global attractor $\mathscr{A}$ of the system (5.6) with Morse decomposition $\mathcal{M} = \{M_1, M_2, M_3\}$. Taking $\mathcal{G} = \mathbb{Z}$, we have

$$\mathfrak{m}_0 = 2, \quad \mathfrak{m}_1 = 1, \quad \mathfrak{m}_2 = \mathfrak{m}_3 = 0.$$

Therefore the formal Poincaré-polynomials are as follows:

$$P_{\mathscr{A}}(t) = \sum_{q=0}^{m}\beta_q t^q \equiv 1, \qquad M_{\mathscr{A}}(t) = \sum_{q=0}^{m}\mathfrak{m}_q t^q = 2 + t,$$

and the equation (5.9) reads

$$(2 + t) - 1 = (1 + t)Q_{\mathscr{A}}(t),$$

from which one also finds that the formal polynomial $Q_{\mathscr{A}}(t) \equiv 1$.

References

[1] J. P. Aubin, *Viability Theory*, Birkhäuser, Boston Basel Berlin, 1991.

[2] J. P. Aubin, *Dynamic Economic Theory*, Springer-Verlag, Berlin Heidelberg, 1997.

[3] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin Heidelberg, 1984.

[4] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston Basel Berlin, 1990.

[5] A. Bressan and C. Delellis, *Existence of optimal strategies for a fire confinement problem*, Commun. Pure Appl. Math. Vol. LXII (2009), no. 6, 0789-0830.

[6] B. Brogliato, *Nonsmooth Mechanics* (2ed.), Springer, London, 1999.

[7] R. Brown, *Generalizations of the Chua equations*, IEEE Trans. Circuits Syst.: Fund. Theory Appl. 40 (1993), 878-883.

[8] T. Caraballo, J. A. Langa and J. Valero, *Asymptotic behaviour of monotone multi-valued dynamical systems*, Dynamical Systems 20 (2005), 301-321.

[9] T. Caraballo, P. E. Kloeden and P. Marín-Rubio, *Global and pullback attractors of setvalued skew product flows*, Ann. Math. Pures Appl. 185 (2006), 523-545.

[10] C. K. Chang, *Critical Point Theory and Its Applications*, Shanghai Science and Technology Press, Shanghai, 1986.

[11] D. Cheban, *Global Attraactors of Set-valued Dynamical and Control Systems*, Nova Science Pub. Inc., 2010.

[12] V. V. Chepyzhov,and M.I. Vishik, *Attractors of Equations of Mathematical Physics*, American Mathematical Society, Providence, RI., 2002.

[13] F. H. Clarke, Yu. S. Ledyaev and R. J. Stern,*Asymptotic stability and smooth Lyapunov functions*. J. Differential Equations 149 (1998), no. 1, 69-114.

[14] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer, 1998.

[15] C. Conley, *Isolated Invariant Sets and the Morse Index*, Regional Conference Series in Mathematics 38, Amer. Math. Soc., Providence RI, 1978.

[16] C. Conley and E. Zehnder, *Morse-type index theory for flows and periodic solutions for Hamiltonian equations*, Commun. Pure Appl. Math. 37 (1984), no. 2, 207-253.

[17] P. Constantin, C. Foias ,B. Nicolaenko and R. Temam, *Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations*, Appl. Math. Sciences 70. Springer, New York-Berlin, 1989.

[18] M. F. Danca and S. Codreanu, *On a possible approximation of discontuous dynamical systems*. Chaos Solitons Fractals 13 (2002), no. 4, 681-691.

[19] K. Deimling, *Multi-valued Differential Equations*, De Gruyter, Berlin, 1998.

[20] M. Fečkan,*Bifurcation from homoclinic to periodic solutions in singularly perturbed differential inclusions*. Proc. Royal Soc. Edinburgh A 127 (1997), no. 12, 727-753.

[21] M. Fečkan, *Chaotic solutions in differential inclusions: Chaos in dry friction problems.* Trans. Amer. Math. Soc. 351 (1999), no. 7, 2861-2873.

[22] A. F. Filippov, *Differential Equations with Discontinuous Righthand Side*. Kluwer Academic Publishers, Dordrecht, 1998.

[23] C. Foias, G. R. Sell and R. Temam, *Inertial Manifolds of Nonlinear Evolutionary Equations*. J. Differential Equations 73 (1988), no. 2, 309-353.

[24] Ch. Glocker, *Set-valued Force Laws, Dynamics of Non-smooth Systems*, Lecture Notes in Appl. Mech. 1, Springer, Berlin, 2001.

[25] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys Monographs 25. AMS Providence, RI, 1998.

[26] S. Hu and N. S. Papageorgiou, *Handbook of set-valued analysis*. Vols. I, II, Kluwer Academic Publishers, Dordrecht, 1999, 2001.

[27] L. Kapitanski and I. Rodnianski *Shape and morse theory of attractors*, Commun. Pure Appl. Math. LIII (2000), no. 2, 218-242.

[28] A. V. Kapustyan and J. Valero, *Attractors of multivalued semiflows generated by differential inclusions and their approximations*, Abstr. Appl. Anal. 5(2000), no. 1, 33-46.

[29] N. N. Krasovskii, *Stability of Motion*, Stanford Univ. Press, Stanford, CA, 1963.

[30] M. Kunze, T. Kupper and Y. Li, *On Conley index theory for non-smooth dynamical systems*, Differential Integral Equations 13 (2000), 479-502.

[31] H. L. Kurland, *The Morse index of an isolated invariant set is a connected simple system.* J. Differential Equations 42 (1981), no. 2, 234-259.

[32] J. Kurzweil, *On the inversion of Ljapunov's second theorem on stability of motion* Amer. Math. Soc. Trans. Ser. II. 24 (1956), 19-77.

[33] O. A. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Lizioni Lincei, Cambridge Univ. Press, Cambridge, New-York, 1991.

[34] J. M. Lasry and R. Robert, *Analyse non-lineaire multivoque* No. 7611, Centre de Recherche de Mathematique de la Decision, Universites de Paris, Dauphine, 1-190.

[35] R. I. Leine R.I. and H. Nijmeijer, *Dynamics and Bifurcations of Non-Smooth Mechanical Systems*, Lecture Notes in Applied and Computational Mechanics 18, Springer, Berlin, 2004.

[36] D. S. Li, *Morse decompositions for general flows and differential inclusions with applications to control systems*. SIAM J. Control Optim. 46 (2007), no. 1, 36-60.

[37] D. S. Li, *On dynamical stability in general dynamical systems*. J. Math. Anal. Appl. 263, no. 2, 455-478 (2001)

[38] D. S. Li and X. X. Zhang, *On the stability in general dynamical systems and differential inclusions*, J. Math. Anal. Appl. 274 (2002), no. 2, 705-724.

[39] D. S. Li and P. E. Kloeden, *Robustness of asymptotic stability to small time delays*. Discrete. Contin. Dyn. Syst. 13 (2005), no. 4, 1007-1034.

[40] Y. Lin, E. D. Sontag, and Y. Wang, *A smooth converse Lyapunov theorem for robust stability*, SIAM J. Control Optim. 34 (1996), no. 1, 124-160.

[41] A. M. Lyapunov, *The general problem of the stability of motion*, Math. Soc. Kharkov. 1892 (Russian). (English Translation: Internat. J. Control 55 (1992), 531-773.

[42] J. L. Massera, *On Liapounov's conditions of stability*, Ann. Math. 50 (1949), 705-721.

[43] V. S. Melnik and J. Valero, *On attractors of multivalued semi-flows and differential inclusions*, Set-Valued Anal. 6 (1998), no. 1, 83-111.

[44] V. S. Melnik and J. Valero, *On global attractors of multivalued semiprocesses and nonautonomous evolution inclusions*. Set-Valued Anal. 8 (2000), 375C403.

[45] J. M. Milnor, *Morse Theory, Annals of Study*. Princeton, 1965.

[46] M. Mrozek, *A cohomological index of Conley type for multivalued admissible flows*, J. Differential Equations 84(1990), 15-51.

[47] M. Rasmussen, *Morse decompositions of nonautonomous dynamical systems*, Trans. Amer. Math. Soc. 359 (2007), no. 10, 5091-5115.

[48] J. W. Robbin, and D. Salamon, *Dynamical systems, shape theory and the Conley index*. Ergodic Theory Dynam. Systems 8 (1988), Charles Conley Memorial Volume, 375-393.

[49] J. C. Robinson, *Infinite-dimensional dynamical systems*, Cambridge University Press, Cambridge, 2001.

[50] K. P. Rybakowski, *The Homotopy Index and Partial Differential Equations*, Springer-Verlag, Berlin Heidelberg, 1987.

[51] K. P. Rybakowski and E. Zehnder *On a Morse equation in Conley's index theory for semiflows on metric spaces*, Ergodic Theory Dynam. Systems 5 (1985), no. 1, 123-143.

[52] José M. R. Sanjurjo, *Shape and Conley index of attractors and isolated invariant sets*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 75, 393-406, Birkhäuser Verlag Basel/Switzerlan, 2007.

[53] G. R. Sell, and Y. C. You, *Dynamics of evolution equations*, Springer-Verlag, New York, 2002.

[54] A. Siconolfi and G. Terrone, *A metric approach to the converse Lyapunov theorem for continuous multivalued dynamics*, Nonlinearity 20 (2007), no. 5, 1077-093.

[55] E. H. Spanier, *Algebraic topology*. McGraw-Hill, New York Toronto London, 1966.

[56] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics* (2 ed.), Springer Verlag, New York, 1997.

[57] A. R. Teel and L. A. Praly, *Smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions*. ESAIM Control Optim. Calc. Var. 5 (2000), 313-367.